The non-symmetric operad pre-Lie is free

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NANTEL BERGERON AND MURIEL LIVERNET

ABSTRACT. We prove that the pre-Lie operad is a free non-symmetric operad.

CONTENTS

INTRODUCTION

Operads are a specific tool for encoding type of algebras. For instance there are operads encoding associative algebras, commutative and associative algebras, Lie algebras, pre-Lie algebras, dendriform algebras, Poisson algebras and so on. A usual way of describing a type of algebras is by giving the generating operations and the relations among them. For instance a Lie algebra L is a vector space together with a bilinear product, the bracket (the generating operation) satisfying the relations $[x, y] = -[y, x]$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. The vector space of all operations one can perform on n distinct variables in a Lie algebra is $\mathcal{L}ie(n)$, the building block of the symmetric operad $\mathcal{L}ie$. Composition in the operad corresponds to composition of operations. The vector space $\mathcal{L}ie(n)$ has a natural action of the symmetric group, so it is a symmetric operad. The case of associative algebras can be considered in two different ways. An associative algebra A is a vector space together with a product satisfying the relation $(xy)z = x(yz)$. The vector space of all operations one can perform on n distinct variables in an associative algebra is $\mathcal{A}s(n)$, the building block of the symmetric operad $\mathcal{A}s$. The vector space $\mathcal{A}s(n)$ has for basis the symmetric group S_n. But, in view of the relation, one can look also at the vector space of all order-preserving operations one can perform on n distinct ordered variables in an associative algebra: this is a vector space of dimension 1 generated by the only operation $x_1 \cdots x_n$. So the non-symmetric operad $\widetilde{\mathcal{A}s}$ describing associative algebras is 1-dimensional for each n: this is the terminal object in the category of non-symmetric operads.

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Here is the connection between symmetric and non-symmetric operads. A symmetric operad $\mathcal{P}$ starts with a graded vector space $(\mathcal{P}(n))_{n\geq 0}$ together with an action of the symmetric group S_n on $\mathcal{P}(n)$ for each n. This data is called a symmetric sequence or an $\mathbb{S}$-module or a vector species. There is a forgetful functor from the category of vector species to the category of graded vector spaces, forgetting the action of the symmetric group. This functor has a left adjoint $\mathcal{S}$ which corresponds to tensoring by the regular representation of the symmetric group. A symmetric (non-symmetric) operad is a monoid in the category of vector species (graded vector spaces). Again there is a forgetful functor from the category of symmetric operads to the category of non-symmetric operads admitting a left adjoint $\mathcal{S}$. The symmetric operad $\mathcal{A}s$ is the image of the non-symmteric operad $\widetilde{\mathcal{A}s}$ by $\mathcal{S}$. It is clear that $\mathcal{L}ie$ is not in the image of $\mathcal{S}$ since the Jacobi relation does not respect the order of the variables $x < y < z$ nor the anti-symmetry relation. Still one can regard $\mathcal{L}ie$ as a non-symmetric operad applying the forgetful functor. Salvatore and Tauraso proved in [5] that the operad $\mathcal{L}ie$ is a free non-symmetric operad.

A free non-symmetric operad describes type of algebras which have a set of generating operations and no relations between them. For instance, magmatic algebras are vector spaces together with a bilinear product. There is a well known free non-symmetric operad, the Stasheff operad, built on Stasheff polytopes, see e.g. [6]. An algebra over the Stasheff operad is a vector space V together with an n-linear product: $V^{\otimes n} \to V$ for each n. From the point of view of homotopy theory, the category of operads is a Quillen category and free operads play an essential role in the homotopy category. One wants to replace an operad $\mathcal{P}$ by a quasi-free resolution, that is, a morphism of operads $\mathcal{Q} \to \mathcal{P}$ where $\mathcal{Q}$ is a free operad endowed with a differential realizing an isomorphism in homology. For instance, a quasi-free resolution of $\widetilde{\mathcal{A}s}$, in the category of non-symmetric operads, is given by the Stasheff operad. Algebras over this operad are A_∞-algebras (associative algebras up to homotopy). This gives us the motivation for studying whether a given symmetric operad is free as a non-symmetric operad or not.

In this paper we prove that the operad pre-Lie is a free non-symmetric operad. Pre-Lie algebras are vector spaces together with a bilinear product satisfying the relation $(x*y)*z-x*(y*z) = (x*z)*y-x*(z*y)$. The operad pre-Lie is based on labelled rooted trees which are of combinatorial interest. In the process of proving the main result, we describe another operad denoted $\mathcal{T}_{\mathrm{Max}}$ also based on rooted trees and having the advantage of being the linearization of an operad in the category of sets. We prove that it is a free non-symmetric operad. The link between the two operads is made via a gradation on labelled rooted trees.

1. The pre-Lie operad and rooted trees

We first recall the definition of the pre-Lie operad based on labelled rooted trees as in [2]. For $n \in \mathbb{N}^*$, the set $\{1,\dots,n\}$ is denoted by $[n]$ and $[0]$ denotes the empty set. The symmetric group on k letters is denoted by S_k. There are many equivalent definitions of operads and we refer to [4] for basics on operads. We work over the ground field $\mathbf{k}$ and vector spaces are considered over $\mathbf{k}$. Here are the definitions needed for the sequel.

Definition 1.1. A (reduced) *non-symmetric* operad is a graded vector space $(\mathcal{P}(n))_{n\geq 1}$, with a unit $1 \in \mathcal{P}(1) = \mathbf{k}$, together with composition maps $\circ_i : \mathcal{P}(n)\otimes\mathcal{P}(m) \to \mathcal{P}(n+m-1)$ for $1 \leq i \leq n$ satisfying the following relations: for $a \in \mathcal{P}(n)$, $b \in \mathcal{P}(m)$ and $c \in \mathcal{P}(\ell)$

$$\begin{array}{lll} (a \circ_i b) \circ_{j+i-1} c & = a \circ_i (b \circ_j c), & \text{for } 1 \leq j \leq m, \\ (a \circ_i b) \circ_j c & = (a \circ_j c) \circ_{i+\ell-1} b, & \text{for } j < i, \\ 1 \circ_1 a & = a, & \\ a \circ_i 1 & = a, & \end{array}$$

A non-trivial composition is a composition $a \circ_i b$ with $a \in \mathcal{P}(n), b \in \mathcal{P}(m)$ and $n, m > 1$.

If in addition each $P(n)$ is acted on the right by the symmetric group S_n and the composition maps are equivariant with respect to this action, then the collection $(\mathcal{P}(n))_n$ forms a *symmetric operad*. An *algebra* over an operad $\mathcal{P}$ is a vector space X endowed with evaluation maps

$$\begin{array}{cccc} ev_n : & \mathcal{P}(n)\otimes X^{\otimes n} & \to & X \\ & p\otimes x_1\otimes\ldots\otimes x_n & \mapsto & p(x_1,\ldots,x_n) \end{array}$$

compatible with the composition maps $\circ_i$: for $p \in \mathcal{P}(n), q \in \mathcal{P}(m), x_i's \in X$ one has

$$(p \circ_i q)(x_1, \ldots, x_{n+m-1}) = p(x_1 \ldots, x_{i-1}, q(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}).$$

If the operad is symmetric the evaluation maps are required to be equivariant with respect to the action of the symmetric group as follows:

$$(p\cdot\sigma)(x_1, \ldots, x_n) = p(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

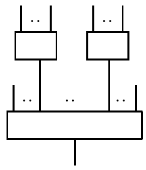
In the sequel an operad will always mean a reduced operad.

Definition 1.2. In this paper we will consider two type of trees: planar rooted trees will represent the composition maps in a non-symmetric operad (see 1.3) and rooted trees will be the objects of our study (see 1.4). Here are the definitions we will use in the sequel.

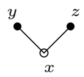
By a (planar) tree we mean a non empty finite connected contractible (planar) graph. All the trees considered are rooted.

In the planar case some edges (*external edges* or *legs*) will have only one adjacent vertex; the other edges are called *internal edges*. There is a distinguished leg called the *root leg*. The other legs are called the leaves. The choice of a root induces a natural orientation of the graph from the leaves to the root. Any vertex has incoming edges and only one outgoing edge. The *arity* of a vertex is the number of incoming edges. A tree with no vertices of arity one is called *reduced*. A planar rooted tree induces a structure of poset on the vertices, where $x < y$ if and only if there is an oriented path in the tree from y to x. Let x be a vertex of a planar rooted tree T. The *full subtree* $T^{(x)}$ of T at x is the subtree of T containing all the vertices $y > x$ and all their adjacent edges. The root leg of $T^{(x)}$ is the half edge with adjacent vertex

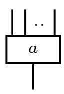
x induced by the unique outgoing edge of x. One represents a planar rooted tree like this:



In the abstract case (non-planar trees) every edge is an internal edge. The *root vertex* will be a distinguished vertex. The choice of a root induces a natural orientation of the graph towards the root. Any vertex has incoming edges and at most one outgoing edge. The other extremity of an incoming (outgoing) edge of the vertex v is called an *incoming (outgoing) vertex of the vertex* v. The root vertex has no outgoing vertex. A rooted tree induces a structure of poset on the vertices, where $x < y$ if and only if there is an oriented path in the tree from y to x. A *leave* is a maximal vertex for this order. The root is the only minimal vertex for this order. Let x be a vertex of a rooted tree T. The *full subtree* $T^{(x)}$ of T derived from the vertex x is the subtree of T containing all the vertices $y > x$. The root of $T^{(x)}$ is x. One represents a rooted tree like this:



Remark 1.3. *Reduced planar tree of operations:* a convenient way to uniquely represent composition of operations in a non-symmetric operad $\mathcal{P}$ is to use a planar rooted tree as in Definition 1.2. An element $a \in \mathcal{P}(n)$ is represented by a planar rooted tree with a single vertex labelled by a with n incoming legs and a single outgoing leg:



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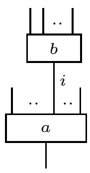
The n leaves are counted from left to right as 1, 2, ..., n. Now if we have $a \in \mathcal{P}(n)$, $b \in \mathcal{P}(m)$ and $1 \leq i \leq n$ we represent the composition $a \circ_i b$ by the planar tree



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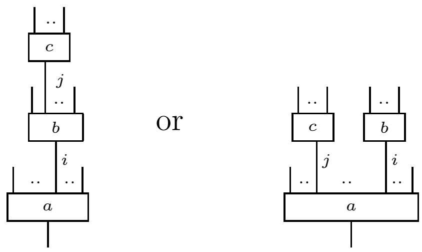
The resulting tree has $n + m - 1$ leaves (counted from left to right) and represents an element of $\mathcal{P}(n + m - 1)$. The two first relations in Definition 1.1 corresponds to the following two trees: for $a \in \mathcal{P}(n)$, $b \in \mathcal{P}(m)$ and $c \in \mathcal{P}(\ell)$ we can have



Each relation is obtained by writing down the two ways of interpreting the tree as a composition of operations. In general a planar tree $\mathbb{T}(a_1, a_2, \ldots, a_k)$ with k vertices labelled by elements $a_i \in \mathcal{P}(n_i)$ where n_i is the number of incoming edges at the i*th* vertex, corresponds to a unique composition of operations in $\mathcal{P}$ independent of any relations.

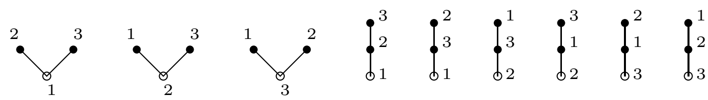
The two last relations in Definition 1.1 say that one can consider reduced trees (no vertices of arity 1) for reduced operads to represent non-trivial composition maps.

Any full subtree of $\mathbb{T}(a_1, a_2, \ldots, a_k)$ is completely determined by the position of its leaves; they form an interval $[p, q]$ where $1 \leq p \leq q \leq n_1 + n_2 + \cdots n_k - k + 1$. A tree *in position* $[p, q]$ will mean the full subtree determined by the position $[p, q]$ of its leaves. If a full subtree in position $[p, q]$ has a single vertex labelled by $a \in \mathcal{P}(n)$ we identify this tree with the element $a \in \mathcal{P}(n)$. It is clear that $n = q - p + 1$.

Two trees of operations $\mathbb{T}(a_1, a_2, \ldots, a_k)$ and $\mathbb{Y}(b_1, b_2, \ldots, b_s)$ are distinct if and only if $\mathbb{T} \neq \mathbb{Y}$ or there exists i such that $a_i \neq b_i$.

Definition 1.4. Let S be a set. An *S-labelled rooted tree* is a non planar rooted tree as in Definition 1.2 whose vertices are in bijection with S. If $S = [n]$, then we talk about n-labelled rooted trees and denote by $\mathcal{T}(n)$ the set of those trees. It is acted on by the symmetric group by permuting the labels.

The set $\mathcal{T}(3)$ has for elements:

(1.1) 

In general $\mathcal{T}(n)$ has n^{n-1} elements (see [1] for more details).

We denote by $\mathbf{k}\mathcal{T}(n)$ the $\mathbf{k}$-vector space spanned by $\mathcal{T}(n)$.

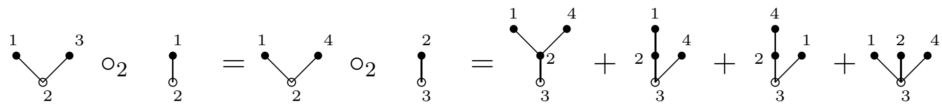
Theorem 1.5. [2, theorem 1.9] *The collection* $(\mathbf{k}\mathcal{T}(n))_{n \geq 1}$ *forms a symmetric operad, the operad pre-Lie denoted by* $\mathcal{PL}$. *Algebras over this operad are pre-Lie algebras, that is, vector spaces* L *together with a product* $*$ *satisfying the relation*

$$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y), \ \forall x, y, z \in L.$$

We recall the operad structure of $\mathcal{PL}$ as explained in [2]. A rooted tree is naturally oriented from the leaves to the root. The set $\mathrm{In}(T, i)$ of incoming vertices of a vertex i is the set of all vertices j such that (j, i) is an edge oriented from j to i. There is also at most one outgoing vertex of a vertex i, i.e. a vertex r such that (i, r) is an oriented edge from i to r, depending whether i is the root of T or not. For $T \in \mathcal{T}(n)$ and $S \in \mathcal{T}(m)$, we define

$$T \circ_i S = \sum_{f:In(T,i)\to[m]} T \circ_i^f S,$$

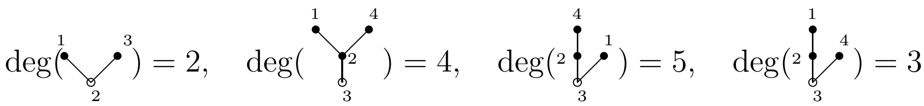
where $T \circ_i^f S$ is the rooted tree obtained by substituting the tree S for the vertex i in T. The outgoing vertex of i, if it exists, becomes the outgoing vertex of the root of S, whereas the incoming vertices of i are grafted on the vertices of S according to the map f. The root of $T \circ_i^f S$ is the root of T if i is not the root of T, and it is the root of S if i is the root of T. There is also a relabelling of the vertices of T and S in $T \circ_i^f S$: we add $i-1$ to the labels of S and $m-1$ to the ones of T which are greater than i. Here is an example:

(1.2) 

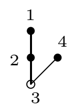
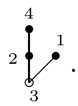
2. A gradation on labelled rooted trees

We introduce a gradation on labelled rooted trees. We prove that in the expansion of the composition of two rooted trees in the operad pre-Lie there is a unique rooted tree of maximal degree and a unique tree of minimal degree, yielding new non-symmetric operad structures on labelled rooted trees.

Definition 2.1. Let T be an n-labelled rooted tree. Let $\{a, b\}$ denote a pair of two adjacent vertices labelled by a and b. The degree of $\{a, b\}$ is $|a - b|$. The *degree of* T denoted by $\deg(T)$ is the sum of the degrees of its pairs of adjacent vertices. For instance



Proposition 2.2. *In the expansion of $T \circ_i S$ in the operad pre-Lie, there is a unique tree of minimal degree and a unique tree of maximal degree.*

For instance, in the equation (1.2) the rooted tree of minimal degree 3 is  and the one of maximal degree 5 is . The other ones are of degree 4.

Proof– Any tree in the expansion of $T \circ_i S$ writes $U_f := T \circ_i^f S$ for some $f : \mathrm{In}(T, i) \to [m]$. To compute the degree of U_f, we compute the degree of a pair of two adjacent vertices $\{a, b\}$ in U_f. There are 4 cases to consider: a) the pair was previously in S or b) it was previously in T and each vertex was different from i, or c) it was in T of the form $\{i, j\}$ for $j \in \mathrm{In}(T, i)$ or d) if i is not the root of T it was of the form $\{i, k\}$ where k is the outgoing vertex of i.

In case a) the degree of the pair in U_f is the same as it was in S.

In case b), let $\{a', b'\}$ be the corresponding pair in T before relabelling. The degree d of the pair $\{a, b\}$ in U_f is the same as the degree d' of $\{a', b'\}$ except if $a' < i < b'$ or $b' < i < a'$, where $d = d' + m - 1$. Let $\mathrm{gap}(T, i)$ be the number of adjacent pairs of vertices in T satisfying the latter condition.

In case c), let $\{i, j\}$ be the pair in T which gives the pair $\{a, b\}$ in U_f. Let d' be the degree of $\{i, j\}$. If $j < i$ then $\{a, b\} = \{f(j) + i - 1, j\}$. Its degree d is minimal and equals d' if $f(j) = 1$. It is maximal and equals $d' + m - 1$ if $f(j) = m$. If $j > i$ then $\{a, b\} = \{f(j) + i - 1, j + m - 1\}$. Its degree d is minimal and equals d' if $f(j) = m$. It is maximal and equals $d' + m - 1$ if $f(j) = 1$.

In case d), let d' be the degree of $\{i, k\}$. If $k < i$ then $\{a, b\} = \{s + i - 1, k\}$ where s is the label of the root of S. It has degree $d' + s - 1$. If $k > i$, then $\{a, b\} = \{s + i - 1, k + m - 1\}$ and has degree $(m - s) + d'$. Let $\epsilon(T, i, s)$ be $0, s - 1, m - s$ according to the different situations, 0 correponding to the one where i is the root of T.

As a conclusion

$$\begin{aligned}(2.1)\quad \deg(T) + \deg(S) + \mathrm{gap}(T, i)(m - 1) + \epsilon(T, i, s) \leq \deg(U_f) \leq \\ \deg(T) + \deg(S) + \mathrm{gap}(T, i)(m - 1) + \epsilon(T, i, s) + |\mathrm{In}(T, i)|(m - 1).\end{aligned}$$

There is a unique f_{Min} such that $\deg(U_{f_{\mathrm{Min}}})$ is minimal and there is a unique f_{Max} such that $\deg(U_{f_{\mathrm{Max}}})$ is maximal:

$$f_{\mathrm{Min}}(k) = \begin{cases} 1 & \text{if } k < i, \\ m & \text{if } k > i, \end{cases} \tag{2.2}$$

$$f_{\mathrm{Max}}(k) = \begin{cases} m & \text{if } k < i, \\ 1 & \text{if } k > i, \end{cases} \tag{2.3}$$

which ends the proof. □

Theorem 2.3. *There are two different non-symmetric operad structures on the collection $(\mathbf{k}\mathcal{T}(n))_{n\geq 1}$ given by the composition maps $T \circ_i^{f_{\mathrm{Min}}} S$ on the one hand and $T \circ_i^{f_{\mathrm{Max}}} S$ on the other hand where f_{Min} and f_{Max} were defined in equations (2.2) and (2.3).*

Proof– A rooted tree T is naturally oriented from its leaves to its root. Any edge is oriented and we denote by (a, b) an edge oriented from the vertex a to the vertex b. Let E_T be the set of the oriented edges of the tree T. For an integer $a \neq i$ we denote by $\tilde{a}_i^m$ the integer a if $a < i$ or $a + m - 1$ if $a > i$. Given a map $f : \mathrm{In}(T, i) \to [m]$, the set $E_{T \circ_i^f S}$ has different type of elements:

- $(a + i - 1, b + i - 1)$ for $(a, b) \in E_S$;
- $(\tilde{a}_i^m, \tilde{b}_i^m)$ for $(a, b) \in E_T$ and $a, b \neq i$;
- $(\tilde{a}_i^m, f(a) + i - 1)$ for $(a, i) \in E_T$;
- $(i + s - 1, \tilde{b}_i^m)$ for $(i, b) \in E_T$.

Let $T \in \mathcal{T}(n), S \in \mathcal{T}(m)$ and $U \in \mathcal{T}(p)$. In order to avoid confusion, we denote by $f^{i,p}_{\mathrm{Max}}$ the map sending $k < i$ to p and $l > i$ to 1. We would like to compare the trees

$$V_1 = (T \circ_i^{f^{i,m}_{\mathrm{Max}}} S) \circ_{j+i-1}^{f^{j+i-1,p}_{\mathrm{Max}}} U \qquad \text{and} \qquad V_2 = T \circ_i^{f^{i,m+p-1}_{\mathrm{Max}}} (S \circ_j^{f^{j,p}_{\mathrm{Max}}} U):$$

- In V_1 and V_2, any $(a, b) \in E_U$ converts to $(a + j + i - 2, b + j + i - 2)$.
- In V_1 and V_2, any $(a, b) \in E_S$ converts to $(\tilde{a}^p_j + i - 1, \tilde{b}^p_j + i - 1)$ if $a, b \neq j$, or converts to $(\tilde{a}^p_j + i - 1, f^{j,p}_{\mathrm{Max}}(a) + i + j - 2)$ if $b = j$ or converts to $(j + i - 1 + u - 1, \tilde{b}^p_j + i - 1)$ if $a = j$.
- In V_1 and V_2, any $(a, b) \in E_T$ with $a, b \neq i$ converts to $(\tilde{a}^{p+m-1}_i, \tilde{b}^{p+m-1}_i)$.
- In V_1 and V_2, any $(a, i) \in E_T$ converts to $(\tilde{a}^{p+m-1}_i, f^{i,m+p-1}_{\mathrm{Max}}(a) + i - 1)$.
- In V_1 and V_2, any $(i, b) \in E_T$ converts to $(i - 1 + \mathrm{root}(S \circ_j U), \tilde{b}^{m+p-1}_i)$, where $\mathrm{root}(S \circ_j U)$ is the root of $S \circ_j U$. More precisely

$$\mathrm{root}(S \circ_j U) = \begin{cases} s & \text{if } s < j \\ u + j - 1 & \text{if } s = j \\ s + p - 1 & \text{if } s > j. \end{cases}$$

The proof of

$$(T \circ_i^{f^{i,m}_{\mathrm{Max}}} S) \circ_j^{f^{j,p}_{\mathrm{Max}}} U = (T \circ_j^{f^{j,p}_{\mathrm{Max}}} U) \circ_{i+p-1}^{f^{i+p-1,m}_{\mathrm{Max}}} S, \text{ for } j < i$$

is similar and left to the reader. So is the proof with f_{Min} instead of f_{Max}. □

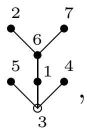
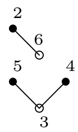
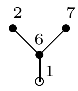
The two operads on labelled rooted trees defined by the theorem are denoted by $\mathcal{T}_{\mathrm{Max}}$ and $\mathcal{T}_{\mathrm{Min}}$. Note that they are linearization of operads in the category of sets. Actually the composition maps are defined at the level of the sets $\mathcal{T}(n)$ and not only at the level of the vector spaces $\mathbf{k}\mathcal{T}(n)$. There is another operad built on rooted trees which has this property: the operad NAP encoding non-asociative permutative algebras in [3], in which f_{NAP} is the constant map with value the root of S. This operad has the advantage of being a symmetric operad.

3. The operad pre-Lie is free as a non-symmetric operad

We show that $\mathcal{T}_{\mathrm{Max}}$ is a free non-symmetric operad. Using Proposition 2.2, we conclude that the operad pre-Lie is free as a non-symmetric operad. To this end we need to introduce some notation on rooted trees.

Definition 3.1. Given two ordered sets S and T, an order-preserving bijection $\phi : S \to T$ induces a natural bijection between the set of S-labelled rooted trees and the set of T-labelled rooted trees also denoted by ϕ. A T-labelled rooted tree X is *isomorphic* to an S-labelled rooted tree Y if $X = \phi(Y)$.

Given a rooted tree $T \in \mathcal{T}(n)$ and a subset $K \subseteq [n]$, we denote by $T\big|_K$ the graph obtained from T by keeping only the vertices of T that are labelled by elements of K and only the edges of T that have two vertices labelled in K. Remark that each connected component of $T\big|_K$ is a rooted tree itself where the root is given by the unique vertex closest to the root of T in the component. Also, for $c \in [n]$ we denote by $T^{(c)}$ the full subtree of T derived from the vertex labelled by c (see Definition 1.2). For example if $K = \{2,3,4,5,6\} \subset [7]$ and

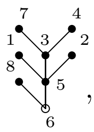
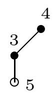
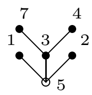
$T =$  , we have $T\big|_K =$  and $T^{(1)} =$  .

For $1 \leq a < b \leq n$, $T \in \mathcal{T}_{\mathrm{Max}}(n-b+a)$ and $S \in \mathcal{T}_{\mathrm{Max}}(b-a+1)$, let $X = T \circ_a S$. Consider the interval $[a,b] = \{a, a+1, \ldots, b\}$, clearly $X\big|_{[a,b]}$ is isomorphic to S under the unique order-preserving bijection $[1, b-a+1] \to [a,b]$. Let $a \leq c \leq b$ be the label of the root of $X\big|_{[a,b]}$. Remark that $X^{(c)}$ is obtained from $X\big|_{[a,b]}$ by grafting subtrees of X at the vertices a and b only. We can then characterize trees X that are obtained from a non-trivial composition $T \circ_a S$ as follows:

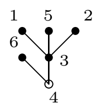
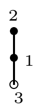
Definition 3.2. A tree $X \in \mathcal{T}_{\mathrm{Max}}(n)$ is called *decomposable* if there exists $1 \leq a < b \leq n$ with $(a,b) \neq (1,n)$ such that

(i) $X\big|_{[a,b]}$ is a rooted tree. Let c be the label of its root. One has $a \leq c \leq b$.
(ii) One has $X^{(c)}|_{[a,b]} = X\big|_{[a,b]}$ and $X^{(c)}$ is obtained from $X\big|_{[a,b]}$ by grafting subtrees of X at the vertices a and b only.
(iii) All subtrees in $X^{(c)} - X\big|_{[a,b]}$ attached at a have their root labelled in $[b+1, n]$.
(iv) All subtrees in $X^{(c)} - X\big|_{[a,b]}$ attached at b have their root labelled in $[1, a-1]$.

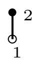
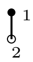
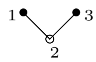
It is clear from the discussion above and the definition of the operad $\mathcal{T}_{\mathrm{Max}}$ that X is decomposable if and only if it is the result of a non-trivial composition. Consequently, we say that X is *indecomposable* if it is not decomposable. That is there is no $1 \leq a < b \leq n$ such that (i)–(iv) are satisfied. For example let

$X =$  , $X\big|_{[3,5]} =$  and $X^{(5)} =$  ,

This tree X is decomposable since for $1 \leq 3 < 5 \leq 8$ we have that $X\big|_{[3,5]}$ is a single tree and the subtrees of $X^{(5)} - X\big|_{[3,5]}$ are attached at 3 and 5 only. Moreover, the subtree attached at 3 has root labelled by $7 \in [6,8]$ and the subtrees attached at 5 have roots labelled by $1, 2 \in [1,2]$. Indeed, in $\mathcal{T}_{\mathrm{Max}}$ we have

$X =$  $\circ_3$  .

The reader may check that the following are all the indecomposable trees of $\mathcal{T}_{\mathrm{Max}}$ up to arity 3:

 ,  and  .

Theorem 3.3. *The non-symmetric operad $\mathcal{T}_{\mathrm{Max}}$ is a free non-symmetric operad.*

Proof. If $\mathcal{T}_{\text{Max}}$ is not free, then for some n there is a tree $X \in \mathcal{T}_{\text{Max}}(n)$ with two distinct constructions from indecomposables. In Remark 1.3, a non-trivial composition of operations is completely determined by a unique reduced planar rooted tree. We then have that $X = \mathbb{T}(T_1, T_2, \ldots, T_r) = \mathbb{Y}(S_1, S_2, \ldots, S_k)$ where $T_1, \ldots, T_r, S_1, \ldots, S_k$ are indecomposables and $\mathbb{T}(T_1, T_2, \ldots, T_r)$ and $\mathbb{Y}(S_1, S_2, \ldots, S_k)$ are two distinct trees of operations in $\mathcal{T}_{\text{Max}}$ with $r, k > 1$.

The tree $X = \mathbb{T}(T_1, T_2, \ldots, T_r)$ is decomposable (by assumption $r \geq 2$). We can find $1 \leq a < b \leq n$, such that $X\big|_{[a,b]}$ is isomorphic to a single T_i in position $[a, b]$ in $\mathbb{T}(T_1, T_2, \ldots, T_r)$. Moreover $X\big|_{[a,b]}$ satisfies (i)–(iv) of Definition 3.2.

If $X\big|_{[a,b]}$ is also isomorphic to a tree S_j in position $[a, b]$ in $\mathbb{Y}(S_1, S_2, \ldots, S_k)$, then we replace X by the smaller tree in $\mathcal{T}_{\text{Max}}(n - b + a)$ that we obtain by removing T_i in $\mathbb{T}(T_1, T_2, \ldots, T_r)$ and removing S_j in $\mathbb{Y}(S_1, S_2, \ldots, S_k)$. Clearly, this new smaller X has two distinct constructions from indecomposables. We can thus assume that $X\big|_{[a,b]}$ is not isomorphic to a single S_j in position $[a, b]$ in $\mathbb{Y}(S_1, S_2, \ldots, S_k)$.

We now study how $X\big|_{[a,b]}$ overlaps in the position $[a, b]$ of $\mathbb{Y}(S_1, S_2, \ldots, S_k)$. Remark first that since all S_j are indecomposables, the interval $[a, b]$ cannot be part of a single S_j of $\mathbb{Y}(S_1, S_2, \ldots, S_k)$. Indeed, that would imply that S_j would contain a subtree satisfying Definition 3.2 which would be a contradiction.

We may assume that $a > 1$. To see this, assume that the only sub-interval $[a, b] \subset [1, n]$ such that $X\big|_{[a,b]}$ is isomorphic to a single T_i in position $[a, b]$ in $\mathbb{T}(T_1, T_2, \ldots, T_r)$ is such that $a = 1$. Assume moreover that the only sub-interval $[a', b'] \subset [1, n]$ such that $X\big|_{[a',b']}$ is isomorphic to a single S_j in position $[a', b']$ in $\mathbb{Y}(S_1, S_2, \ldots, S_k)$ is such that $a' = 1$. Since S_j is indecomposable, we must have $b > b'$. Similarly, since T_i is indecomposable, we must have $b < b'$. This implies that $b = b'$ and $T_i = S_j$. This possibility was excluded above. So we must have $a > 1$ or $a' > 1$. In the case where $a = 1$ and $a' > 1$ we could just interchange the role of $\mathbb{T}(T_1, T_2, \ldots, T_r)$ and $\mathbb{Y}(S_1, S_2, \ldots, S_k)$ and assume that we have $a > 1$.

Now, since T_i is indecomposable, there is no subinterval $[c, d] \subseteq [a, b]$ such that $X\big|_{[c,d]}$ is isomorphic to a full subtree of operations $\mathbb{Y}'(S_{j_1}, S_{j_2}, \ldots, S_{j_\ell})$. Assume we can find $c < a \leq d < b$ such that $X\big|_{[c,d]} \cong \mathbb{Y}'(S_{j_1}, S_{j_2}, \ldots, S_{j_\ell})$ satisfies the Definition 3.2.

The graph $X\big|_{[a,d]}$ is contained in the trees $X\big|_{[a,b]}$ and $X\big|_{[c,d]}$. Let e be the label of the root of $X\big|_{[a,b]}$ and f be the label of the root of $X\big|_{[c,d]}$. The two full subtrees $X^{(e)}$ and $X^{(f)}$ both contain $X\big|_{[a,d]}$. This implies that either $X^{(f)}$ is fully contained in $X^{(e)}$, or $X^{(e)}$ is fully contained in $X^{(f)}$.

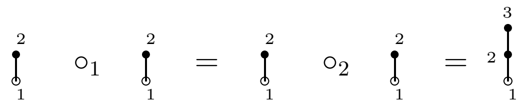
Let us assume that $X^{(f)}$ is fully contained in $X^{(e)}$, that means $X\big|_{[a,b]}$ and $X\big|_{[c,d]}$ are both subtrees of $X^{(e)}$. From Definition 3.2, we know that $X^{(e)}$ is obtained from $X\big|_{[a,b]}$ by graphting subtrees of X at the vertices a and b only. The vertex c is in $X^{(e)}$ but not in $X\big|_{[a,b]}$. It is part of a subtree attached to a or b. Since c is part of a subtree with root f one has $f \notin]a, b[$. The vertex f is a (can not be b since $f \leq d$) or is attached to a or b. If f is attached to b then there is a path $c \to f \to b$. The tree $X|_{[c,d]}$ has its root labelled by f so there is a path $d \to f$. The tree $X|_{[a,b]}$ contains the vertices b and d and any path from d to b so there is a path $d \to f \to b$ in $X|_{[a,b]}$. Hence $f = a$ for $f \notin]a, b]$. As a conclusion c is part of a subtree attached to a. By (iii) of Definition 3.2 applied to the tree $X\Big|_{[a,b]}$, the subtree must have a root $r \in [b+1, n]$. This is a contradiction, the root r is part of any path joining a and c and $r \notin [c, d]$, hence not in $X\Big|_{[c,d]}$. The case where $X^{(e)}$ is fully contained in $X^{(f)}$ is argued similarly, using condition (iv) of Definition 3.2, and leads to a contradiction as well.

The same argument holds in case we can find $a < c \leq b < d$.

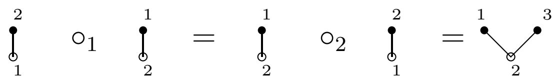
The only case remaining is that the interval $[p, q]$ associated to any full subtree $\mathbb{Y}(S_1, \dots, S_k)^{(S_j)}$ of $\mathbb{Y}(S_1, \dots, S_k)$, satisfies $[a, b] \cap [p, q] = \emptyset$ or $[a, b] \subset [p, q]$. There is at least one interval satisfying $[a, b] \subset [p, q]$ (take the full tree $\mathbb{Y}(S_1, \dots, S_k)$ and $[p, q] = [1, n]$). Let $[p, q]$ be the smallest interval such that $[a, b] \subset [p, q]$ and let $\mathbb{Y}(S_1, \dots, S_k)^{(S_j)} = \mathbb{Y}'(S_{i_1}, \dots, S_{i_l})$ be the full subtree it determines. Its root is labelled by S_j. The interval $[u, v]$ associated to any proper full subtree of $\mathbb{Y}'(S_{i_1}, \dots, S_{i_l})$ satisfies $[a, b] \cap [u, v] = \emptyset$. Consequently $X|_{[a,b]}$ is isomorphic to $S_j|_{[\alpha,\beta]}$ for some interval $[\alpha, \beta]$ isomorphic to $[a, b]$. This is impossible since X satisfies the conditions of Definition 3.2 and S_j is indecomposable.

We must conclude that $\mathcal{T}_{\mathrm{Max}}$ is free. □

Remark 3.4. The non-symmetric operads $\mathcal{T}_{\mathrm{Min}}$ and NAP are not free. Indeed, in the operad $\mathcal{T}_{\mathrm{Min}}$ one has the following relation:



And in the operad NAP one has the following relation



Remark 3.5. Let $\mathbf{k}\mathcal{T}^0_{\mathrm{Max}}(n)$ denote the $\mathbf{k}$-vector space spanned by the indecomposables of $\mathcal{T}_{\mathrm{Max}}(n)$ $(n > 1)$ and let β_n be its dimension. Let $\alpha(x) = \sum_{n\geq 1} \alpha_n x^n$ be the Hilbert series associated to the free non-symmetric operad generated by the vector spaces $\mathbf{k}\mathcal{T}^0_{\mathrm{Max}}(n)$. It is well known (see e.g. [5]) that one has the identity

$$\beta(\alpha(x)) + x = \alpha(x),$$

where $\beta(x) = \sum_{n\geq 2} \beta_n x^n$. Theorem 3.3 implies that $\alpha_n = n^{n-1}$. As a consequence, we get that the Hilbert series for indecomposable of $\mathcal{T}_{\mathrm{Max}}$ is

$$\mathcal{H}_{\mathcal{T}^0_{\mathrm{Max}}}(x) = \sum_{n\geq 2} \dim\left(\mathbf{k}\mathcal{T}^0_{\mathrm{Max}}(n)\right) x^n = 2x^2 + x^3 + 14x^4 + 146x^5 + \\ +1994x^6 + 32853x^7 + 630320x^8 + 13759430x^9 + \cdots .$$

Corollary 3.6. *The non-symmetric operad pre-Lie is a free non-symmetric operad.*

Proof. Let $\mathcal{F}$ be the free non-symmetric operad on indecomposable trees. By the universal property of $\mathcal{F}$, there is a unique morphism of operads

$$\phi : \mathcal{F} \to \mathcal{PL}$$

extending the inclusion of indecomposable trees in $\mathcal{PL}$. We prove that this map is surjective by induction on the degree of a tree. Trees of degree 1 are indecomposables (see Definition 3.2). Let $t \in \mathcal{PL}(n)$ be a tree of degree $k \geq n-1$. If t is indecomposable then $t = \phi(t)$. If t is decomposable there are trees $u \in \mathcal{PL}(r), v \in \mathcal{PL}(s)$, with $r, s < n$ such that $t = u \circ_i^{f_{\mathrm{Max}}} v$ in $\mathcal{T}_{\mathrm{Max}}$. By Proposition 2.2 one has in $\mathcal{PL}$

$$u \circ_i v = t + \sum_j t_j$$

where $t_j \in \mathcal{PL}(n)$ has degree $k_j < k$. From equation (2.1) we deduce that the degrees of u and v are also lower than k. By induction, the trees u, v and t_j's are in the image of ϕ, so is t. Thus, the operad morphism ϕ is surjective. Theorem 3.3 implies that the vector spaces $\mathcal{F}(n)$ and $\mathcal{PL}(n)$ have the same dimension, thus the operad morphism ϕ is an isomorphism. □

Remark 3.7. The Hilbert Series for the free non-symmetric operad on indecomposables and the operad $\mathcal{PL}$ are the same as in Remark 3.5.

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